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Morita Contexts and their Lattices of Relations

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We study the interaction between the lattices of relations of members of a general Morita context. The pairs of reversing-order maps are defined, which determine the dualities between the lattices of 'closed' relations. Under rather weak conditions, these dualities can be composed obtaining the projectivities defined by simple maps.

KEY WORDS: lattices of relations; general Morita context; duality; projectivity.
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1. INTRODUCTION AND PRELIMINARIES

The notion of Morita equivalence is fundamental in both Algebra and Analysis. The machinery of Morita equivalence is in terms of projective modules, tensor products, and *Hom* functors. Bass (1962) developed the notion of a Morita context, which facilitates the application of the theory.

Morita equivalence was adapted to C*-algebras by Rieffel in the 1970s (Rieffel, 1974) and has since become a standard tool for analyzing group C*-algebras and crossed products. Our treatment of lattices of 'closed' relations of a general Morita context reflects our emphasis on Morita equivalence. We originally intended to phrase all our results and calculation in terms of Morita equivalence for quantales as in Paseka (2002). However, as we progressed, we found that it is more convenient to compute in the setting of semigroups that generalizes the viewpoints of the ring theory, the C*-algebra theory and the theory of quantales.

In this paper, we present an interaction between the lattices of relations of members of a general Morita context. The presented results were inspired by a similar situation for Morita contexts between rings (see Kashu, 1998).

The paper is organized as follows: Section 1 provides a brief introduction to the notion of general Morita context and defines the investigated dualities and

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projectivities. Section 2 is devoted to show how the dualities and projectivities are related. Section 3 shortly translates the results of Section 2 into the setting of the lattice of all subacts (submodules as in Kashu (1998)).

For a general overview of the Morita equivalence theory over semigroups we refer, for example to Talwar (1996), for facts concerning quantales and quantale modules, in general we refer to Rosenthal (1990).

Let R and S be semigroups, P an R, S-biact, Q an S, R-biact such that there are biact maps $(-,-) : P \times Q \longrightarrow R$ and $[-,-] : Q \times P \longrightarrow S$ such that $(x \cdot s, y) = (x, s \cdot y)$, $[y \cdot r, x] = [y, r \cdot x]$ (that is, these maps are *balanced*), $(x_1, y) \cdot x_2 = x_1 \cdot [y, x_2]$, $[y_1, x] \cdot y_2 = y_1 \cdot (x, y_2)$ for each $x, x_1, x_2 \in P$, $y, y_1, y_2 \in Q$, $s \in S, r \in R$.

The 6-tuple $(R, S, P, Q, (-,-), [-,-])$ is called a *general Morita context* and the pair (P, Q) a *general Morita pair*.

A general Morita context $(R, S, P, Q, (-,-), [-,-])$ is said to be a *general quantale Morita context* if R, S are quantales, P is an R, S-bimodule, Q is an S, R-bimodule and $(-,-)$ and $[-,-]$ preserve arbitrary joins in each variable.

Let $h : A \times B \to C$ be an arbitrary map, A, B, C sets. There exists a unique map $\overline{h} : \mathcal{P}(A \times A) \times \mathcal{P}(B \times B) \to \mathcal{P}(C \times C)$ which is sup-preserving in each variable and which satisfies $\overline{h}(\{(a_1, a_2)\}, \{(b_1, b_2)\}) = \{(h(a_1, b_1), h(a_2, b_2)\})$ for all $a_1, a_2 \in A$ and all $b_1, b_2 \in B$. This map is called the *relational extension of* h.

Moreover, since the actions $\overline{h}(-, Y) : \mathcal{P}(A \times A) \to \mathcal{P}(C \times C)$ and $\overline{h}(X, -) : \mathcal{P}(B \times B) \to \mathcal{P}(C \times C)$ are sup-preserving for all $X \subseteq A \times A$ and all $Y \subseteq B \times B$, they have adjoints denoted $Y \xrightarrow{h}_l -$ and $X \xrightarrow{h}_r -$, respectively. Note that, for all $Z \subseteq C \times C$, $\overline{h}(X, Y) \subseteq Z$ iff $X \subseteq Y \xrightarrow{h}_l Z$ iff $Y \subseteq X \xrightarrow{h}_r Z$.

Proposition 1. *Let* $(R, S, P, Q, (-,-), [-,-])$ *be a* general Morita context. *Then,* $(\mathcal{P}(R \times R), \mathcal{P}(S \times S), \mathcal{P}(P \times P), \mathcal{P}(Q \times Q), \overline{(-,-)}, \overline{[-,-]})$ *is a* general quantale Morita context*; here* $\mathcal{P}(R \times R), \mathcal{P}(S \times S)$ *are quantales with respect to the induced operations* $\overline{\cdot_R}$, $\overline{\cdot_S}$ *and arbitrary unions,* $\mathcal{P}(P \times P)$ *is an* $\mathcal{P}(R \times R), \mathcal{P}(S \times S)$*-bimodule with respect to the induced left and right actions and arbitrary unions,* $\mathcal{P}(Q \times Q)$ *is an* $\mathcal{P}(S \times S)$, $\mathcal{P}(R \times R)$*-bimodule with respect to the induced left and right actions and arbitrary unions.*

Proof: It is evident since $R \times R$ and $S \times S$ are semigroups, $P \times P$ an $R \times R, S \times S$-biact, $Q \times Q$ an $S \times S, R \times R$-biact, $\mathcal{P}(R \times R)$ is a free quantale over $R \times R$, $\mathcal{P}(S \times S)$ is a free quantale over $S \times S$, $\mathcal{P}(P \times P)$ is a free $\mathcal{P}(R \times R), \mathcal{P}(S \times S)$ quantale bimodule over $P \times P$ and $\mathcal{P}(Q \times Q)$ is a free $\mathcal{P}(S \times S), \mathcal{P}(R \times R)$ quantale bimodule over $Q \times Q$. □

For an arbitrary general Morita context $(R, S, P, Q, (-,-), [-,-])$ we consider the following pairs of maps between the lattices of relations:

$$
\begin{array}{llll}
\alpha_P(\sigma) &= \mathrm{id}_P \overset{\cdot_P}{\longrightarrow}_r \sigma, & \beta_P(\psi) &= \mathrm{id}_Q \overset{[-,-]}{\longrightarrow}_r \psi,\\
\alpha_Q(\tau) &= \mathrm{id}_Q \overset{\cdot_Q}{\longrightarrow}_l \tau, & \beta_Q(\psi) &= \mathrm{id}_Q \overset{[-,-]}{\longrightarrow}_l \psi,\\
\gamma_P(\sigma) &= \overline{[\mathrm{id}_Q, \sigma]}, & \delta_P(\psi) &= \mathrm{id}_P \overline{\cdot_P}\, \psi,\\
\gamma_Q(\tau) &= \overline{(\tau, \mathrm{id}_P)}, & \delta_Q(\psi) &= \psi \overline{\cdot_Q}\, \mathrm{id}_Q,\\
r_S(\psi) &= \psi \overset{\cdot_S}{\longrightarrow}_r \mathrm{id}_S, & l_S(\psi) &= \psi \overset{\cdot_S}{\longrightarrow}_l \mathrm{id}_S,\\
r_P(\sigma) &= \sigma \overset{(-,-)}{\longrightarrow}_r \mathrm{id}_S, & l_P(\tau) &= \tau \overset{(-,-)}{\longrightarrow}_l \mathrm{id}_S,\\
G_P(\sigma) &= \sigma \overset{\cdot_P}{\longrightarrow}_r \mathrm{id}_P, & H_P(\psi) &= \psi \overset{\cdot_P}{\longrightarrow}_l \mathrm{id}_P,\\
G_Q(\tau) &= \tau \overset{\cdot_Q}{\longrightarrow}_l \mathrm{id}_Q, & H_Q(\psi) &= \psi \overset{\cdot_Q}{\longrightarrow}_r \mathrm{id}_Q;
\end{array}
$$

here $\psi \in \mathcal{P}(S \times S), \sigma \in \mathcal{P}(P \times P), \tau \in \mathcal{P}(Q \times Q)$;

$$
\begin{tikzcd}
\mathcal{P}(P \times P) \arrow[r, "r_P"] \arrow[d, shift left, "\gamma_P"] \arrow[dr, "G_P"] & \mathcal{P}(Q \times Q) \arrow[r, "l_P"] \arrow[d, shift left, "\alpha_Q"] & \mathcal{P}(P \times P) \arrow[d, shift left, "\alpha_P"] \\
\mathcal{P}(S \times S) \arrow[r, "r_S"] \arrow[u, shift left, "\delta_P"] \arrow[dr, "H_Q"] & \mathcal{P}(S \times S) \arrow[r, "l_S"] \arrow[u, shift left, "\beta_Q"] \arrow[ur, "H_P"] \arrow[d, shift left, "\delta_Q"] & \mathcal{P}(S \times S) \arrow[u, shift left, "\beta_P"] \\
 & \mathcal{P}(Q \times Q). \arrow[u, shift left, "\gamma_Q"] \arrow[ur, "G_Q"] &
\end{tikzcd}
$$

Lemma 2. *The pairs of maps $(r_S, l_S), (r_P, l_P), (G_P, H_P), (G_Q, H_Q)$ form Galois connections.*

Proof: Let $\psi, \psi_1, \psi_2 \in \mathcal{P}(S \times S)$, $\sigma \in \mathcal{P}(P \times P)$, $\tau \in \mathcal{P}(Q \times Q)$. We have $\psi_2 \subseteq r_S(\psi_1)$ iff $\psi_2 \subseteq \psi_1 \overset{\cdot_S}{\longrightarrow}_r \mathrm{id}_S$ iff $\psi_1 \overline{\cdot_S} \psi_2 \subseteq \mathrm{id}_S$ iff $\psi_1 \subseteq \psi_2 \overset{\cdot_S}{\longrightarrow}_l \mathrm{id}_S$ iff $\psi_1 \subseteq l_S(\psi_2)$. Similarly, $\tau \subseteq r_P(\sigma)$ iff $\tau \subseteq \sigma \overset{[-,-]}{\longrightarrow}_r \mathrm{id}_S$ iff $\overline{(\sigma, \tau)} \subseteq \mathrm{id}_S$ iff $\sigma \subseteq \tau \overset{[-,-]}{\longrightarrow}_l \mathrm{id}_S$ iff $\sigma \subseteq l_P(\tau)$. Moreover, $\sigma \subseteq H_P(\psi)$ iff $\sigma \subseteq \psi \overset{\cdot_P}{\longrightarrow}_l \mathrm{id}_P$ iff $\sigma \overline{\cdot_P} \psi \subseteq \mathrm{id}_P$ iff $\psi \subseteq \sigma \overset{\cdot_P}{\longrightarrow}_r \mathrm{id}_P$ iff $\psi \subseteq G_P(\sigma)$. □

Note that the Lemma 2 implies that we have $\psi \subseteq r_S(l_S(\psi))$, $\psi \subseteq l_S(r_S(\psi))$, $\tau \subseteq r_P(l_P(\tau))$, $\sigma \subseteq l_P(r_P(\sigma))$, $\sigma \subseteq H_P(G_P(\sigma))$, $\psi \subseteq G_P(H_P(\psi))$, $\tau \subseteq H_Q(G_Q(\tau))$ and $\psi \subseteq G_Q(H_Q(\psi))$ for all $\psi \in \mathcal{P}(S \times S)$, $\sigma \in \mathcal{P}(P \times P)$, $\tau \in \mathcal{P}(Q \times Q)$.

For any map $f : A \to B$, we shall denote $\mathrm{Im} f$ the set $\{y : y = f(x), x \in A\}$. Then $\mathrm{Im}(r_S) = \{\psi \in \mathcal{P}(S \times S) : \psi = r_S(l_S(\psi))\}$, $\mathrm{Im}(l_S) = \{\psi \in \mathcal{P}(S \times S) : \psi = l_S(r_S(\psi))\}$, $\mathrm{Im}(r_P) = \{\tau \in \mathcal{P}(Q \times Q) : \tau = r_P(l_P(\tau))\}$, $\mathrm{Im}(l_P) = \{\sigma \in \mathcal{P}(P \times P) : \sigma = l_P(r_P(\sigma))\}$, $\mathrm{Im}(H_P) = \{\sigma \in \mathcal{P}(P \times P) : \sigma = H_P(G_P(\sigma))\}$, $\mathrm{Im}(G_P) = \{\psi \in \mathcal{P}(S \times S) : \psi = G_P(H_P(\psi))\}$, $\mathrm{Im}(H_Q) = \{\tau \in \mathcal{P}(Q \times Q) : \sigma = H_Q(G_Q(\tau))\}$ and $\mathrm{Im}(G_Q) = \{\psi \in \mathcal{P}(S \times S) : \psi = G_Q(H_Q(\psi))\}$ are closure systems and we obtain the following dualities between them:

$$\mathcal{P}(S \times S) \supseteq \mathrm{Im}(l_S) \underset{l_S}{\overset{r_S}{\rightleftarrows}} \mathrm{Im}(r_S) \subseteq \mathcal{P}(S \times S) \qquad (D1),$$

$$\mathcal{P}(P \times P) \supseteq \mathrm{Im}(l_P) \underset{l_P}{\overset{r_P}{\rightleftarrows}} \mathrm{Im}(r_P) \subseteq \mathcal{P}(Q \times Q) \qquad (D2),$$

$$\mathcal{P}(P \times P) \supseteq \mathrm{Im}(H_P) \underset{H_P}{\overset{G_P}{\rightleftarrows}} \mathrm{Im}(G_P) \subseteq \mathcal{P}(S \times S) \qquad (D3),$$

$$\mathcal{P}(Q \times Q) \supseteq \mathrm{Im}(H_Q) \underset{H_Q}{\overset{G_Q}{\rightleftarrows}} \mathrm{Im}(G_Q) \subseteq \mathcal{P}(S \times S) \qquad (D4).$$

Recall that the right S-act P is *faithful* for S (left R-act P is *faithful* for R) whenever $p \cdot s_1 = p \cdot s_2$ for all $p \in P$ implies $s_1 = s_2$ ($r_1 \cdot p = r_2 \cdot p$ for all $p \in P$ implies $r_1 = r_2$). The preceding condition is clearly equivalent to the condition

$$\text{(A) } \mathrm{id}_P \overline{\cdot_P} \psi \subseteq \mathrm{id}_P \Longrightarrow \psi \subseteq \mathrm{id}_S \quad [\ \text{(A}'\text{) } \mu \overline{\cdot_P}\, \mathrm{id}_P \subseteq \mathrm{id}_P \Longrightarrow \mu \subseteq \mathrm{id}_R\ ]$$

for all $\psi \in \mathcal{P}(S \times S)$ ($\mu \in \mathcal{P}(R \times R)$). In particular, the left S-act Q is *faithful* for S (right R-act Q is *faithful* for R) whenever the following condition is satisfied:

$$\text{(C) } \psi \overline{\cdot_Q} \mathrm{id}_Q \subseteq \mathrm{id}_Q \Longrightarrow \psi \subseteq \mathrm{id}_S \quad [\ \text{(C}'\text{) } \mathrm{id}_Q \overline{\cdot_Q} \mu \subseteq \mathrm{id}_Q \Longrightarrow \mu \subseteq \mathrm{id}_R\ ]$$

for all $\psi \in \mathcal{P}(S \times S)$ ($\mu \in \mathcal{P}(R \times R)$).

Similarly, the S, R-biact Q is *right faithful* (*left faithful*) for the R, S-biact P whenever $[q, p_1] = [q, p_2]$ ($(p_1, q) = (p_2, q)$) for all $q \in Q$ implies $p_1 = p_2$. This condition is equivalent to the condition

$$\text{(B) } \overline{[\mathrm{id}_Q, \sigma]} \subseteq \mathrm{id}_S \Longrightarrow \sigma \subseteq \mathrm{id}_P \quad [\ \text{(B}'\text{) } \overline{(\sigma, \mathrm{id}_Q)} \subseteq \mathrm{id}_S \Longrightarrow \sigma \subseteq \mathrm{id}_P\ ]$$

for all $\sigma \in \mathcal{P}(P \times P)$. In particular, the R, S-biact P is *left faithful* (*right faithful*) for the S, R-biact Q whenever the following condition is satisfied:

$$\text{(D) } \overline{[\tau, \mathrm{id}_P]} \subseteq \mathrm{id}_S \Longrightarrow \tau \subseteq \mathrm{id}_Q \quad [\ \text{(D}'\text{) } \overline{(\mathrm{id}_P, \tau)} \subseteq \mathrm{id}_S \Longrightarrow \tau \subseteq \mathrm{id}_Q\ ]$$

for all $\tau \in \mathcal{P}(Q \times Q)$.

A general Morita context is said to be *nondegenerate* if the conditions (A), (B), (A′), (B′), (C), (D), (C′) and (D′) are satisfied.

2. THE COMPOSITION OF DUALITIES ON LATTICES OF RELATIONS

To compose the dualities (D1) and (D3), we need the equality $\mathrm{Im}(r_S) = \mathrm{Im}(G_P)$. This can be obtained in the case when the maps r_S and G_P are expressed by one another. Thus, we have the following lemma.

Lemma 3. *Condition (A) implies the equality $r_S = G_P \circ \delta_P$. Condition (B) implies the equality $G_P = r_S \circ \gamma_P$. Both conditions (A) and (B) ensure the equality $Im(r_S) = Im(G_P)$ and so the dualities (D1) and (D3) can be composed obtaining the projectivity*

$$\mathcal{P}(P \times P) \supseteq Im(H_P) \underset{H_P \circ r_S}{\overset{l_S \circ G_P}{\rightleftarrows}} Im(l_S) \subseteq \mathcal{P}(S \times S) \qquad (P1)$$

Proof: Let $\psi, \varphi \in \mathcal{P}(S \times S)$. We have $r_S(\psi) \subseteq G_P(\delta_P(\psi))$. Namely, $\psi \overline{\cdot_S} \varphi \subseteq \mathrm{id}_S$ implies $\delta_P(\psi) \overline{\cdot_S} \varphi = (\mathrm{id}_P \overline{\cdot_P} \psi) \overline{\cdot_S} \varphi = \mathrm{id}_P \overline{\cdot_P} (\psi \overline{\cdot_S} \varphi) \subseteq \mathrm{id}_P \overline{\cdot_P} \mathrm{id}_S \subseteq \mathrm{id}_P$. Hence, putting $\varphi = r_S(\psi)$, we get the required inclusion. Conversely, we have $r_S(\psi) \supseteq G_P(\delta_P(\psi))$. This follows from the fact that $\varphi \subseteq G_P(\delta_P(\psi))$ is equivalent to $\mathrm{id}_P \overline{\cdot_P} \psi \overline{\cdot_S} \varphi \subseteq \mathrm{id}_P$. Using the faithfulness of P we get $\psi \overline{\cdot_S} \varphi \subseteq \mathrm{id}_S$, hence $\varphi \subseteq r_S(\psi)$. In particular, $\mathrm{Im}(r_S) \subseteq \mathrm{Im}(G_P)$.

Similarly, let us check that $G_P = r_S \circ \gamma_P$. Assume that $\varphi \subseteq G_P(\sigma)$. Then $\sigma \overline{\cdot_P} \varphi \subseteq \mathrm{id}_P$. Note that $r_S(\gamma_P(\sigma)) = r_S(\overline{[\mathrm{id}_Q, \sigma]})$. Then, $\varphi \subseteq r_S(\gamma_P(\sigma))$ iff $\overline{[\mathrm{id}_Q, \sigma]} \overline{\cdot_S} \varphi \subseteq \mathrm{id}_S$. We have $\overline{[\mathrm{id}_Q, \sigma]} \overline{\cdot_S} \varphi = \overline{[\mathrm{id}_Q, \sigma \overline{\cdot_P} \varphi]} \subseteq \overline{[\mathrm{id}_Q, \mathrm{id}_P]} \subseteq \mathrm{id}_S$. Conversely, let $\varphi \subseteq r_S(\gamma_P(\sigma))$. Then $\overline{[\mathrm{id}_Q, \sigma \overline{\cdot_P} \varphi]} \subseteq \mathrm{id}_S$. By the faithfulness of Q we have that $\sigma \overline{\cdot_P} \varphi \subseteq \mathrm{id}_P$. Hence, $\varphi \subseteq G_P(\sigma)$. In particular, $\mathrm{Im}(r_S) \supseteq \mathrm{Im}(G_P)$. □

Lemma 4. *Condition (A) implies the equality $l_S = \alpha_P \circ H_P$, therefore $\alpha_P(\sigma) = (l_S \circ G_P)(\sigma)$ for every $\sigma \in Im(H_P)$. Condition (B) implies the equality $H_P = \beta_P \circ l_S$, therefore $\beta(\psi) = (H_P \circ r_S)(\psi)$ for every $\psi \in Im(l_S)$.*

Proof: As in the previous lemma, $\varphi \subseteq l_S(\psi)$ iff $\varphi \overline{\cdot_S} \psi \subseteq \mathrm{Id}_S$ and $\varphi \subseteq \alpha_P(H_P(\psi))$ iff $\mathrm{Id}_P \overline{\cdot_P} \varphi \subseteq H_P(\psi)$ iff $\mathrm{Id}_P \overline{\cdot_P} \varphi \overline{\cdot_P} \psi \subseteq \mathrm{Id}_P$.

Assuming $\varphi \subseteq l_S(\psi)$, we get immediately that $\mathrm{Id}_P \overline{\cdot_P} \varphi \overline{\cdot_P} \psi \subseteq \mathrm{Id}_P$. Conversely, let $\varphi \subseteq l_S(\psi)$. By the faithfulness of P we get that $\varphi \overline{\cdot_S} \psi \subseteq \mathrm{Id}_S$. Hence, $\varphi \subseteq l_S(\psi)$. Alltogether, $l_S = \alpha_P \circ H_P$.

Similarly, let $\sigma \subseteq H_P(\psi)$ iff $\sigma \overline{\cdot_P} \psi \subseteq \mathrm{Id}_P$ and $\sigma \subseteq \beta_P(l_S(\psi))$ iff $\overline{[\mathrm{id}_Q, \sigma]} \subseteq l_S(\psi)$ iff $\overline{[\mathrm{id}_Q, \sigma]} \overline{\cdot_S} \varphi \subseteq \mathrm{id}_S$ iff $\overline{[\mathrm{id}_Q, \sigma \overline{\cdot_P} \varphi]} \subseteq \mathrm{id}_S$. Consequently, $\sigma \subseteq H_P(\psi)$ implies that $\overline{[\mathrm{id}_Q, \sigma]} \overline{\cdot_P} \varphi \subseteq \mathrm{id}_S$. Conversely, let $\sigma \subseteq \beta_P(l_S(\psi))$. Applying the faithfulness of Q we get that $\sigma \overline{\cdot_P} \psi \subseteq \mathrm{Id}_P$. Hence, $\sigma \subseteq H_P(\psi)$ and it follows that $H_P = \beta_P \circ l_S$. The both considerations, in turn, imply (by the Galois connection property) the remaining part of the lemma. □

Theorem 5. *If conditions (A) and (B) are satisfied then dualities (D1) and (D3) can be composed and the resulting projectivity (P1) is determined by the maps α_P and β_P:*

$$\begin{tikzcd}
Im(H_P) \arrow[r, shift left, "G_P"] \arrow[dr, shift right, "\alpha_P"'] & Im(G_P) = Im(r_S) \arrow[l, shift left, "H_P"] \arrow[d, shift left, "l_S"] \\
 & Im(l_S) \arrow[ul, shift right, "\beta_P"'] \arrow[u, shift left, "r_S"]
\end{tikzcd}$$

In a similar vein we can prove the following (dual) statements:

Lemma 6. *Condition (A′) implies the equality $l_P = H_P \circ \gamma_P$. Condition (B′) implies the equality $H_P = l_P \circ \delta_Q$. Both conditions (A′) and (B′) ensure the equality $Im(l_P) = Im(H_P)$, and so the dualities (D2) and (D3) can be composed obtaining the projectivity*

$$\mathcal{P}(Q \times Q) \supseteq Im(r_P) \underset{r_P \circ H_P}{\overset{G_P \circ l_P}{\rightleftarrows}} Im(G_P) \subseteq \mathcal{P}(S \times S) \qquad (P2)$$

Lemma 7. *Condition (A′) implies the equality $r_P = \beta_Q \circ G_P$, therefore $\beta_Q(\psi) = (r_P \circ H_P)(\psi)$ for every $\psi \in Im(G_P)$. Condition (B′) implies the equality $G_P = \alpha_Q \circ r_P$, therefore $\alpha_Q(\tau) = (G_P \circ l_P)(\tau)$ for every $\tau \in Im(r_P)$.*

Theorem 8. *If conditions (A′) and (B′) are satisfied, then dualities (D2) and (D3) can be composed and the resulting projectivity (P2) is determined by the maps α_Q and β_Q:*

$$\begin{tikzcd}
Im(r_P) \arrow[r, shift left, "l_P"] \arrow[dr, shift right, "\alpha_Q"'] & Im(l_P) = Im(H_P) \arrow[l, shift left, "r_P"] \arrow[d, shift left, "G_P"] \\
 & Im(G_P) \arrow[ul, shift right, "\beta_Q"'] \arrow[u, shift left, "H_P"]
\end{tikzcd}$$

Lemma 9. *Condition (C) implies the equality* $l_S = G_Q \circ \delta_Q$. *Condition (D) implies the equality* $G_Q = l_S \circ \gamma_P$. *Both conditions (C) and (D) ensure the equality* $Im(l_S) = Im(G_Q)$ *and so the dualities (D1) and (D4) can be composed obtaining the projectivity*

$$\mathcal{P}(Q \times Q) \supseteq Im(H_Q) \underset{H_Q \circ l_S}{\overset{r_S \circ G_Q}{\rightleftarrows}} Im(r_S) \subseteq \mathcal{P}(S \times S) \qquad (P4)$$

Lemma 10. *Condition (C) implies the equality* $r_S = \alpha_Q \circ H_Q$, *therefore* $\alpha_Q(\tau) = (r_S \circ G_Q)(\tau)$ *for every* $\tau \in Im(H_Q)$. *Condition (D) implies the equality* $H_Q = \beta_Q \circ r_S$, *therefore* $\beta_Q(\psi) = (H_Q \circ l_S)(\psi)$ *for every* $\psi \in Im(r_S)$.

Theorem 11. *If conditions (C) and (D) are satisfied, then dualities (D1) and (D4) can be composed and the resulting projectivity (P3) is determined by the maps* α_Q *and* β_Q:

$$\begin{tikzcd}
Im(H_Q) \arrow[r, shift left, "G_Q"] \arrow[dr, shift right, "\alpha_Q"'] & Im(G_Q) = Im(l_S) \arrow[l, shift left, "H_Q"] \arrow[d, shift left, "r_S"] \\
& Im(r_S) \arrow[u, shift left, "l_S"] \arrow[ul, shift right, "\beta_Q"']
\end{tikzcd}$$

Lemma 12. *Condition (C′) implies the equality* $H_Q = r_P \circ \delta_P$. *Condition (D′) implies the equality* $G_Q = l_S \circ \gamma_P$. *Both conditions (C) and (D) ensure the equality* $Im(r_P) = Im(H_Q)$ *and so the dualities (D2) and (D4) can be composed obtaining the projectivity*

$$\mathcal{P}(P \times P) \supseteq Im(l_P) \underset{l_P \circ H_Q}{\overset{G_Q \circ r_P}{\rightleftarrows}} Im(G_Q) \subseteq \mathcal{P}(S \times S) \qquad (P3)$$

Lemma 13. *Condition (C′) implies the equality* $l_P = \beta_P \circ G_Q$, *therefore* $\beta_P(\psi) = (l_P \circ H_Q)(\psi)$ *for every* $\psi \in Im(G_Q)$. *Condition (D′) implies the equality* $G_Q = \alpha_P \circ l_P$, *therefore* $\alpha_P(\sigma) = (G_Q \circ r_P)(\sigma)$ *for every* $\sigma \in Im(l_P)$.

Theorem 14. *If conditions (C′) and (D′) are satisfied, then dualities (D2) and (D4) can be composed and the resulting projectivity (P4) is determined by the* *maps α_P and β_P:*

$$
\begin{tikzcd}
Im(l_P) \arrow[r, shift left, "r_P"] \arrow[dr, shift right, "\alpha_P"'] & Im(r_P) = Im(H_P) \arrow[l, shift left, "l_P"] \arrow[d, shift left, "G_Q"] \\
& Im(G_Q) \arrow[u, shift left, "H_Q"] \arrow[ul, shift right, "\beta_P"']
\end{tikzcd}
$$

Combining Theorems 5, 8, 11, and 14, the principal result of this note is

Theorem 15. *If the context $(R, S, P, Q, (-, -), [-, -])$ is nondegenerate, then we have the following situation:*

$$
\begin{tikzcd}
Im(l_P) = Im(H_P) \arrow[r, shift left, "r_P"] \arrow[d, shift left, "\alpha_P"] \arrow[dr, "G_P"] & Im(r_P) = Im(H_Q) \arrow[l, shift left, "l_P"] \arrow[d, shift left, "\alpha_Q"] \arrow[dl, "G_Q"'] \\
Im(l_S) = Im(G_Q) \arrow[r, shift left, "r_S"] \arrow[u, shift left, "\beta_P"] \arrow[ur, shift right, "H_Q"'] & Im(r_S) = Im(G_P), \arrow[l, shift left, "l_S"] \arrow[u, shift left, "\beta_Q"] \arrow[ul, shift left, "H_P"]
\end{tikzcd}
$$

where all maps are bijections: four compatible dualities (r_S, l_S), (G_P, H_P), (r_P, l_P), (G_Q, H_Q), and two projectivities (α_P, β_P), (α_Q, β_Q) which are the compositions of respective dualities.

3. THE COMPOSITION OF DUALITIES FOR SUBACTS

We now proceed to look shortly at the case of subacts and factorsets. Recall that each subset X (factorset Y) of some S-act T corresponds to the identity subrelation id_X of id_T (to some equivalence overrelation of id_T). Similarly, a right (left) subact X of some right (left) S-act T corresponds to the identity subrelation id_X of id_T such that $\mathrm{id}_X \overline{\cdot_T} \mathrm{id}_S \subseteq \mathrm{id}_X$ ($\mathrm{id}_S \overline{\cdot_T} \mathrm{id}_X \subseteq \mathrm{id}_X$). A corresponding notion for factorsets is a right (left) congruence on id_T. Analogous considerations to the following one can be done for any suitable notion of a module over some "good" algebraic structure (rings, semirings, quantales, C*-algebras etc.).

We can then investigate the reversing-order maps between the complete lattices of right subacts $\mathbf{L}(P_S)$ and $\mathbf{L}(S_S)$ and the complete lattices of left congruences $\mathbf{Con}({}_SS)$ and $\mathbf{Con}({}_SQ)$.

Note that, for a nondegenerate context,

$$\begin{array}{ll}
r_S(\mathbf{L}(S_S)) \subseteq \mathbf{Con}({}_SS), & l_S(\mathbf{Con}({}_SS)) \subseteq \mathbf{L}(S_S), \\
r_P(\mathbf{L}(P_S)) \subseteq \mathbf{Con}({}_SQ), & l_P(\mathbf{Con}({}_SQ)) \subseteq \mathbf{L}(P_S), \\
\alpha_P(\mathbf{L}(P_S)) \subseteq \mathbf{L}(S_S), & \beta_P(\mathbf{L}(S_S)) \subseteq \mathbf{L}(P_S), \\
\gamma_P(\mathbf{L}(P_S)) \subseteq \mathbf{L}(S_S), & \delta_P(\mathbf{L}(S_S)) \subseteq \mathbf{L}(P_S), \\
\alpha_Q(\mathbf{Con}({}_SQ)) \subseteq \mathbf{Con}({}_SS), & \beta_Q(\mathbf{Con}({}_SS)) \subseteq \mathbf{Con}({}_SQ), \\
\gamma_Q(\mathbf{Con}({}_SQ)) \subseteq \mathbf{Con}({}_SS), & \delta_Q(\mathbf{Con}({}_SS)) \subseteq \mathbf{Con}({}_SQ), \\
G_P(\mathbf{L}(P_S)) \subseteq \mathbf{Con}({}_SS), & H_P(\mathbf{Con}({}_SS)) \subseteq \mathbf{L}(P_S), \\
G_Q(\mathbf{Con}({}_SQ)) \subseteq \mathbf{L}(S_S), & H_Q(\mathbf{L}(S_S)) \subseteq \mathbf{Con}({}_SQ).
\end{array}$$

For any map μ between the lattices of all relations, we shall define a restriction μ^s to the lattices of right subacts and a restriction μ^c to the lattices of left congruences. Then the pairs of maps (r_S^s, l_S^c), (r_P^s, l_P^c), (G_P^s, H_P^c), (G_Q^c, H_Q^s) form Galois connections and all the statements from Section 2 can be translated to the subacts-congruences setting.

In particular, the following theorem holds

Theorem 16. *If the context* $(R, S, P, Q, (-,-), [-,-])$ *is nondegenerate then we have the following situation:*

$$\begin{tikzcd}
Im(l_P^c) = Im(H_P^c) \arrow[r, shift left, "r_P^s"] \arrow[d, shift left, "\alpha_P^s"] \arrow[dr, shift left, "G_P^s"] & Im(r_P^s) = Im(H_Q^s) \arrow[l, shift left, "l_P^c"] \arrow[d, shift left, "\alpha_Q^c"] \arrow[dl, shift left, "G_Q^c"] \\
Im(l_S^c) = Im(G_Q^c) \arrow[r, shift left, "r_S^s"] \arrow[u, shift left, "\beta_P^s"] \arrow[ur, shift left, "H_Q^s"] & Im(r_S^s) = Im(G_P^s), \arrow[l, shift left, "l_S^c"] \arrow[u, shift left, "\beta_Q^c"] \arrow[ul, shift left, "H_P^c"]
\end{tikzcd}$$

where all maps are bijections: four compatible dualities (r_S^s, l_S^c), (G_P^s, H_P^c), (r_P^s, l_P^c), (G_Q^c, H_Q^s), *and two projectivities* (α_P^s, β_P^s), (α_Q^c, β_Q^c) *which are the compositions of respective dualities.*

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